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QUANTUM GROUP COVARIANT SYSTEMS

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Abstract

The meaning of quantum group transformation properties is discussed in some detail by comparing the (co)actions of the quantum group with those of the corresponding Lie group, both of which have the same algebraic (matrix) form of the transformation. Various algebras are considered which are covariant with respect to the quantum (super) groups $SU_q(2)$, $SU_q(1,1)$, $SU_q(1|1)$, $SU_q(n)$, $SU_q(m|n)$, $OSP_q(1|2)$ as well as deformed Minkowski space-time algebras.

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1 Introduction

The transformation properties of physical systems related to the Lie groups are of great importance for the understanding of Nature. As a result, applications of the Lie group theory take place in quite different branches of physics and the corresponding formalism is very well developed. The quantum groups and quantum algebras extracted from the quantum inverse scattering method (QISM) happen to be quite similar or even richer mathematical objects as compared to Lie groups and Lie algebras.

In this paper we shall point out some peculiarities of the quantum group interpretations when the formal transformations of the physical quantities coincide with the usual ones while the coefficients (elements) of these transformations are now non-commutative quantities belonging to a quantum group (QG) or a quantum algebra.

These objects (QG and q-algebras) are described using the language of Hopf algebras. In the general situation of Lie group theory one has the Lie algebra $Lie(G)$ (or better to say the corresponding universal enveloping algebra) with non-commutative multiplication and symmetric coproduct Δ , and the commutative algebra of functions F on the Lie group manifold G with non-symmetric coproduct $\Delta : F \rightarrow F \otimes F$. After a q-deformation (or "quantization") the corresponding objects $Lie_q(G)$ and F_q start to be much more similar in between as the Hopf algebras with non-commutative multiplications and non-symmetric coproducts in both cases. Hence, it looks natural to have the same physical interpretation for transformations including both of them.

Putting aside the complicated integrable models solved by the quantum inverse scattering method and the quantum conformal field theory we consider systems with finite degrees of freedom such as a set of q-oscillators \mathcal{A} covariant under the coaction φ of the quantum (super-) group F_q . When the coaction $\varphi : \mathcal{A} \rightarrow F_q \otimes \mathcal{A}$ does not preserve the physical observables such as Hamiltonian, momenta, etc, the standard problem of the tensor product decomposition of $\mathcal{H}_F \otimes \mathcal{H}_\mathcal{A}$ emerges, where \mathcal{H}_F and $\mathcal{H}_\mathcal{A}$ are the state spaces of the corresponding algebras. If instead of F_q we have the quantum algebra $Lie_q(G)$ then its representations are almost identical to the undeformed algebra (let us omit plenty of technicalities related with the case when q is a root of unity). However, the deformed algebra of functions F_q is a new algebraic object with its own representation theory and the corresponding decomposition problem of $\mathcal{H}_F \otimes \mathcal{H}_\mathcal{A}$ or $\mathcal{H}_F \otimes \mathcal{H}_F$ is new as well.

2 Covariant systems

2.1. \mathcal{A}_q as $su_q(2)$ -covariant algebra.

Let us start with a simple covariant system. The q -oscillator algebra \mathcal{A}_q has three generators with commutation relations (for q real one has $A = q^{N/2}a = q^N\alpha$)

$$aa^\dagger - qa^\dagger a = q^{-N}, \quad [N, a] = -a, \quad [N, a^\dagger] = a^\dagger; \quad (1)$$

$$AA^\dagger - q^2 A^\dagger A = 1, \quad [N, A] = -A, \quad [N, A^\dagger] = A^\dagger; \quad (2)$$

$$[\alpha, \alpha^\dagger] = q^{-2N}, \quad [N, \alpha] = -\alpha, \quad [N, \alpha^\dagger] = \alpha^\dagger. \quad (3)$$

The third set can be obtained from the quantum algebra $su_q(2)$ (with generators X_+, X_-, J and well-known commutation relations and coproduct Δ) by a contraction procedure with fixed q [8] ($\lambda = q - q^{-1}$)

$$\alpha = \lim \lambda^{1/2} X_+ / q^s, \quad s \rightarrow \infty, \quad N = s - J.$$

From this contraction one could find that although there is no natural coproduct for \mathcal{A}_q , the formulas for the $su_q(2)$ coproduct that survive under the contraction procedure could be interpreted as covariance of the algebra \mathcal{A}_q with respect to the quantum algebra $su_q(2)$. The corresponding map is $\psi : \mathcal{A}_q \rightarrow \mathcal{A}_q \otimes su_q(2)$, such that

$$\psi(N) = N - J,$$

$$\psi(\alpha) = \alpha q^{-J} + \sqrt{\lambda} q^{-N} X_+, \quad (4)$$

$$\psi(\alpha^\dagger) = \alpha^\dagger q^{-J} + \sqrt{\lambda} q^{-N} X_-.$$

It is easy to check the following consistency properties for this coaction : $(\psi \otimes id) \circ \psi = (id \otimes \Delta) \circ \psi$ and $(id \otimes \epsilon) \circ \psi = id$ as well as that ψ preserves the defining relations of \mathcal{A}_q . However, the central element z of the algebra \mathcal{A}_q

$$z = \alpha^\dagger \alpha - [N; q^{-2}], \quad [x; q] = (1 - q^x)/(1 - q), \quad (5)$$

is not invariant under this coaction: $\psi(z) \neq z$.

If we choose the Hamiltonian of the q -oscillator as $H = \alpha^\dagger \alpha$ and restrict ourselves to the irreducible representation \mathcal{H}_F of \mathcal{A}_q with the vacuum state: $\alpha|0\rangle = 0, N|0\rangle = 0$ (for $q \in (0, 1)$ there are other irreps [5, 8]) then $z = 0$ and

$$H = \alpha^\dagger \alpha = [N; q^{-2}] = (1 - q^{-2N})/(1 - q^{-2}). \quad (6)$$

The spectrum of H and its eigenstates are obvious

$$spec H = \{[n; q^{-2}], \quad n = 0, 1, 2, \dots\},$$

$$|n\rangle = ([n; q^{-2}]!)^{-1/2} (\alpha^\dagger)^n |0\rangle.$$

After the coaction the changed Hamiltonian describes an "interaction" of the q -oscillator with the q -spin

$$H_I = \alpha^\dagger \alpha q^{-2J} + \lambda q^{-2N} X_- X_+ + \sqrt{\lambda} q^{-N-J} (X_- \alpha / q + q \alpha^* X_+) . \quad (7)$$

It acts in the space of physical states $\mathcal{H}_{ph} = \mathcal{H}_F \otimes V_j$, where V_j is an irreducible finite dimensional representation of the $su_q(2)$ of dimension $2j+1$. This space is decomposed into the direct sum of $2j+1$ irreducible representations of \mathcal{A}_q : $\mathcal{H}_{ph} = \sum_k \mathcal{H}_F^{(k)}$ with corresponding vacuum states $|0\rangle_k$, $k = 0, 1, \dots, 2j$

$$|0\rangle_k = \sum_{m=0}^k |m\rangle \otimes |j-k+m; j\rangle = c(m, k) , \quad (8)$$

where

$$|m\rangle \in \mathcal{H}_F , \quad |l; j\rangle \in V_j , \quad J|l; j\rangle = l|l; j\rangle ,$$

$$X_+ |m; j\rangle = N_+(m, j) |m+1; j\rangle , \quad N_+^2(m, j) = [j-m]_q [j+m+1]_q ,$$

$$c(m, k) = (-\sqrt{\lambda} q^{j-k+1})^m ([m; q^{-2}]!)^{-1/2} \prod_{l=0}^{m-1} N_+(j-k+l, j) .$$

The spectrum of H_I coincides with that of H up to the multiplicative factor $q^{2(j-k)}$ in each subspace $\mathcal{H}_F^{(k)}$, but it has the multiplicity $2j+1$

$$\text{spec } H = \{q^{2(j-k)} [n; q^{-2}] , \quad n = 0, 1, 2, \dots\} .$$

The central element $\psi(z)$ has $2j+1$ eigenvalues $-[k-j; q^{-2}]$.

It is interesting to point out that this coaction has no classical (non-deformed) counterpart in the limit $q \rightarrow 1$ in the quantum theory. Such a limit exists in the Poisson-Lie theory (see e.g. [22]). The connection of the q -oscillator algebra \mathcal{A}_q with $su_q(2)$ through the contraction procedure gives rise also to a more complicated coaction of the quantum group $SU_q(2)$ on \mathcal{A}_q (see Subsec. 2.5 and 2.6).

2.2. \mathcal{A}_q as $SU_q(1, 1)$ -covariant algebra.

Let us consider the second set (2) of the q -oscillator algebra \mathcal{A}_q generators with relation

$$AA^\dagger = qA^\dagger A + 1 .$$

redenoting q^2 by q and putting $q \in (0, 1)$. This relation reminds us of the quantum plane with $xy = qyx$ a central extension. Using the two component column $X^t = (A, A^\dagger)$ it can be rewritten in the R -matrix form [7, 10]

$$\hat{R}X \otimes X = qX \otimes X - q^{-1}J , \quad (9)$$

where $\hat{R} = \mathcal{P}R$ is the R -matrix of $su_q(2)$ and J is the four component column $J^t = (0, 1, -q, 0)$ obviously related to the well-known q -metric 2×2 matrix

ϵ_q . This relation is preserved under the transformation $\psi : X \rightarrow \psi(X) = TX$, with T being the 2×2 matrix of the quantum group $SU_q(1, 1)$ generators

$$T = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix}$$

which satisfies the FRT-relation $\hat{R}T \otimes T = T \otimes T\hat{R}$ [1]. The invariance of the inhomogeneous term is just another form of the q -metric relation

$$T\epsilon_q T^t = \epsilon_q \det_q T, \quad T \otimes TJ = T_1 T_2 J = \det_q TJ, \quad (10)$$

provided that the quantum determinant of T is 1: $\det_q T = aa^* - qbb^* = a^*a - b^*b/q = 1$ (a defining condition for $SU_q(1, 1)$). (One can consider central extension of the real quantum plane as well with $|q| = 1$ covariant with respect to $SL_q(2, \mathbf{R})$. Then the reality condition will fix the phase of the constant term.) The map $\psi : \mathcal{A}_q \rightarrow \mathcal{A}_q \otimes SU_q(1, 1)$ or in terms of the generators

$$\psi(A) = aA + bA^\dagger, \quad \psi(A^\dagger) = a^*A^\dagger + b^*A \quad (11)$$

satisfies all properties of a coaction. Its form is reminiscent of the famous Bogoliubov transformation. However, now the "coefficients" are non-commuting. The q -oscillator Hamiltonian acting in the same space \mathcal{H}_F as (6)

$$H = A^\dagger A = [N; q] = (q^N - 1)/(q - 1) \quad (12)$$

(it differs from the previous one by q^N factor and renotation of q^2) is also not invariant under the coaction ψ

$$\psi(H) = A^\dagger A + b^*b + [2]_q A^\dagger A b^*b + b^*a A^2 + a^*b(A^\dagger)^2. \quad (13)$$

The commutation relations of the $SU_q(1, 1)$ generators (as well as those of $SU_q(2)$)

$$ab = qba, \quad ab^* = qb^*a, \quad bb^* = b^*b, \\ [a, a^*] = \lambda b^*b, \quad \lambda = (q - 1/q),$$

themselves remind us of the q -oscillator algebra ($a \sim \alpha^\dagger, a^* \sim \alpha, b \sim b^* \sim q^{-N}$) with the additional condition due to $\det_q T = 1$

$$aa^* = 1 + qb^*b, \quad a^*a = 1 + b^*b/q.$$

The deformation parameter q being less than 1 forces us to consider the irreducible representation of $SU_q(1, 1)$ in the Hilbert space $l_2(\mathbf{Z})$ with basis $|m\rangle, m = \dots, -2, -1, 0, 1, 2, \dots$ which consists of the eigenstates of commuting operators b, b^* for which a acts as a creation (shift) operator

$$b|m\rangle = e^{i\phi} q^{-m} |m\rangle, \quad a|m\rangle = c_m |m+1\rangle, \quad a^*|m\rangle = c_{m-1} |m-1\rangle,$$

$$c_m^2 = 1 + q^{-2m-1}.$$

Hence the transformed Hamiltonian is defined in the space $\mathcal{H}_F \otimes l_2(\mathbf{Z})$. It has the same spectrum $\{[n; q], n = 0, 1, \dots\}$ with infinite multiplicity. The corresponding vacuum states are

$$|0 \rangle_{(\psi)}^{(k)} = \sum_{j=0}^{\infty} (-1)^j x_j |2j \rangle \otimes |k-j \rangle, \quad (14)$$

$$x_j = ([2j-1; q]!!/[2j; q]!!)^{1/2} \Pi_{i=1}^j q^{i-k}/c_{k-i},$$

where the second vector in the tensor product belongs to the $SU_q(1,1)$ irrep space $l_2(\mathbf{Z})$.

As in the case of the representation theory the *invariant subspaces* of the QG F (coaction) corepresentation V can be defined as $W \subset V$ such that $\phi : W \rightarrow F \otimes W$. The *invariant elements* of the F -corepresentation V do not change at all: $\phi(v) = v$ (or better to write $1_F \otimes v$ for one has the possibility in the corresponding representation of the dual Hopf algebra $(F)^*$ to contract a dual element $X \in (F)^*$ with 1_F to get a number $X(1_F)$).

The extensions of the previous examples to higher rank quantum groups give rise to covariant algebras corresponding to different quantum homogeneous spaces [23], systems of (super) q -oscillators [5, 4, 6, 15] and examples of non-commutative geometry.

2.3. The covariant super- q -oscillator algebra $s\mathcal{A}_q$ [4] refers to the quantum super-group $SU_q(1|1)$, with the T -matrix of the generators

$$T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix}$$

and the commutation relations

$$\begin{aligned} a\beta &= q\beta a, & a\gamma &= q\gamma a, \\ d\beta &= q\beta d, & d\gamma &= q\gamma d, \end{aligned}$$

$$\beta\gamma = -\gamma\beta, \quad \beta^2 = \gamma^2 = 0, \quad [a, d] = (q - 1/q)\gamma\beta.$$

Fixing the central element (super-determinant) $s\det_q T = (a - \beta \frac{1}{d} \gamma)/d = 1$ one gets a simple relation between the even generators d and a

$$d = a - \beta\gamma/qd = a - \beta\gamma/qa$$

due to the nilpotency of the odd generators. The involution ($*$ -operation) is introduced as follows:

$$d = 1/a^*, \quad \gamma = d\beta^*d = a\beta^*a.$$

This involution leads to $TT^\dagger = 1$ and it is consistent with the Z_2 -grading. One has for the generators a, a^*, β, β^*

$$a^*\beta = \beta a^*/q, \quad [a, a^*] = (1 - q^2)\beta^*\beta, \quad \beta\beta^* = -q^2\beta^*\beta,$$

$$aa^* = 1 + \beta\beta^*, \quad a^*a = 1 - \beta^*\beta.$$

Introducing $a' = a(1 + \beta^*\beta/2) = (1 - \beta\beta^*/2)a$, one gets $a'a'^* = a'^*a' = 1$ and the factorization of the T -matrix ($a' = 1/a'^*$)

$$T = \begin{pmatrix} a' & 0 \\ 0 & 1/a'^* \end{pmatrix} \begin{pmatrix} (1 - \beta^*\beta/2) & \beta/qa' \\ \beta^*/a'^* & (1 + \beta^*\beta/2) \end{pmatrix}$$

with unit super-determinant. One concludes that the q -deformation (quantization) of the $SU(1|1)$ super-group is realized by the unitary scaling operator Λ , $\Lambda^* = \Lambda^{-1}$ acting on the Grassmann variables η and η^* , which are not quantized $\beta = \Lambda\eta$,

$$\Lambda\beta = q\beta\Lambda, \quad \Lambda\beta^* = q\beta^*\Lambda, \quad \Lambda 1 = 1$$

and $a' = \exp(i\varphi)\Lambda$. Hence, like in the non-deformed case the representations of the $SU_q(1|1)$ are parametrized by the phase and the Grassmann variable.

The corresponding $SU_q(1|1)$ covariant system of the super- q -oscillator $s\mathcal{A}_q$ has four generators $A, A^\dagger, B, B^\dagger$ with relations [4]

$$\begin{aligned} AA^\dagger - q^2 A^\dagger A &= 1, \quad BB^\dagger + B^\dagger B = 1 + (q^2 - 1)A^\dagger A, \\ AB &= qBA, \quad AB^\dagger = qB^\dagger A, \quad B^2 = B^{\dagger 2} = 0. \end{aligned}$$

Using the R -matrix formalism similar to (9) it is not difficult to show that these relations as well as the Hamiltonian $H = A^\dagger A + B^\dagger B$ are invariant w.r.t. the coaction

$$\begin{pmatrix} A \\ B \end{pmatrix} \rightarrow \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix}.$$

However, one can consider Hamiltonians which are not invariant w.r.t. the QG transformation. The latter one will extend the initial system after the transformation. In particular, one can consider different versions of the q -deformed $N = 2$ -SUSY-algebra taken as the super-charges

$$Q = A^\dagger B, \quad Q^\dagger = B^\dagger A \quad \text{or} \quad Q = A^\dagger f, \quad Q^\dagger = A f^\dagger,$$

where f and f^\dagger are free fermions $f = q^{-N}B$ commuting with A, A^\dagger [4]. In all the cases the coaction does not extend the space of states for the representation theory of $SU_q(1|1)$ is rather poor and only additional Grassmann parameters appear after the coaction.

2.4 $SU_q(n)$ - and $SU_q(m|n)$ -covariant (super) algebras.

Let us introduce $2n$ generating elements of the $SU_q(n)$ -covariant oscillator algebra $\mathcal{A}_q(n)$ [5], written as n -component column and row vectors

$$A^t = (A_1, \dots, A_n), \quad A^\dagger = (A_1^\dagger, \dots, A_n^\dagger).$$

Their commutation relations in the R -matrix form (a spectral parameter independent Zamolodchikov - Faddeev algebra) [7] ($\hat{R}_{\mathcal{P}} = \mathcal{P}\hat{R}^t\mathcal{P}$)

$$\hat{R}A \otimes A = qA \otimes A, \quad A^\dagger \otimes A^\dagger \hat{R}_{\mathcal{P}} = qA^\dagger \otimes A^\dagger,$$

$$A \otimes A^\dagger = qA_2^\dagger \hat{R}A_2 + I$$

demonstrate easily that, due to the FRT-relation $[\hat{R}, T_1 T_2] = 0$ and $TT^\dagger = I$ the coaction $\phi(A) = TA$, $\phi(A^\dagger) = A^\dagger T^\dagger$ satisfies all the requirements.

One can rewrite these relations in the form (14) using the $2n$ -component vector $X = (A_1, \dots, A_n; A_1^\dagger, \dots, A_n^\dagger)$ and the corresponding $2n \times 2n$ matrix R , which happens to be the R -matrix of the quantum group $Sp_q(2n)$. Then the

inhomogeneous term ($2n$ -component vector J) is expressed using the invariant matrix C of $Sp_q(2n)$: $T^t C T = C$ [1, 10].

The invariant Hamiltonian w.r.t. the $SU_q(n)$ -coaction is

$$H = A_1^\dagger A_1 + A_2^\dagger A_2 + \dots + A_n^\dagger A_n ,$$

which in the Fock space can be written in terms of the mode number operators N_k , $k = 1, 2, \dots$

$$H = (q^{2(N_1+N_2+\dots+N_n)} - 1)/(q^2 - 1) .$$

It was already pointed out that the $SU_q(1,1)$ quantum group is related to the q -oscillator algebra. The same is true for the $SU_q(2)$: its defining relations coincide with (3) up to notations and some factors. A realization of the $SU_q(n)$ requires $n(n-1)/2$ q -oscillators and $n-1$ phase factors [14]. Hence, transforming the algebra $\mathcal{A}_q(n) \rightarrow SU_q(n) \otimes \mathcal{A}_q(n)$ one jumps from the n degrees of freedom to the $n(n+1)/2$ ones.

The situation for the quantum super-group $SU_q(m|n)$ is similar. The co-variant super-algebra $s\text{-}\mathcal{A}_q(m|n)$ has m boson and n fermion mutually non-commuting q -oscillators [4] or vice versa. The realization of the $SU_q(m|n)$ requires $m(m-1)/2 + n(n-1)/2$ q -oscillators and $m \times n$ Grassmann parameters as well as some phase factors.

2.5 The next example of a covariant system is related to the reflection equation algebra \mathcal{K} (or the q -Minkowski space-time algebra, or the quantum sphere algebra) (see e.g. [18]). Its quantum group covariance depends on the set of R -matrices in the defining equation (a reflection equation)

$$R_{12}^{(1)} K_1 R_{12}^{(2)} K_2 = K_2 R_{12}^{(3)} K_1 R_{12}^{(4)} , \quad (15)$$

with the coaction $\varphi(K) = K' = TKS$ where $R^{(j)}$, $j = 1, \dots, 4$ define the commutation relations of the T and S entries [18]. For the simple $SU_q(2)$ covariant case one has $R_{12}^{(1)} = R_{12}^{(3)} = R_{12}$, $R_{12}^{(2)} = R_{12}^{(4)} = R_{21}$ with the $sl_q(2)$ matrix R and $T = S^\dagger = S^{-1}$. K is the following 2×2 matrix of generators

$$K = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} .$$

So the algebra \mathcal{K} has four generators: $\alpha, \beta, \gamma, \delta$ with relations

$$\begin{aligned} \alpha\beta &= q^{-2}\beta\alpha, & [\delta, \beta] &= q^{-1}\lambda\alpha\beta, \\ \alpha\gamma &= q^2\gamma\alpha, & [\delta, \gamma] &= -q^{-1}\lambda\gamma\alpha, \\ [\alpha, \delta] &= 0, & [\beta, \gamma] &= q^{-1}\lambda(\delta - \alpha)\alpha, \end{aligned}$$

and two central elements

$$c_1 = q^{-1}\alpha + q\delta, \quad c_2 = \alpha\delta - q^2\beta\gamma.$$

One has the covariance of \mathcal{K} with respect to the quantum group $SU_q(2)$ with the coaction $\varphi : \mathcal{K} \rightarrow SU_q(2) \otimes \mathcal{K}$ which is easy to write using the matrix form

$$\varphi(K) = K' = UKU^\dagger,$$

where $U = (U^\dagger)^{-1}$ is the following 2×2 matrix of the $SU_q(2)$ generators

$$U = \begin{pmatrix} a & qb \\ -b^\dagger & a^\dagger \end{pmatrix}.$$

Due to the fact that the q-determinant is equal to one

$$aa^\dagger + q^2b^\dagger b = a^\dagger a + b^\dagger b = 1$$

the quantum group $SU_q(2)$ has essentially one unitary irreducible representation \mathcal{H}_F [23] with vacuum state $|0\rangle$: $a|0\rangle = 0$, $b|0\rangle = 1|0\rangle$,

$$|n\rangle = (a^\dagger)^n |0\rangle / \Pi_j c_j, \quad c_n^2 = (1 - q^{2n}).$$

The algebra \mathcal{K} with the $*$ -operation $K = K^\dagger$ has many irreducible representations [19]. Let us fix one of them \mathcal{H}_1 then after the coaction φ the transformed algebra K' generated by $(K')_{ij} = \varphi(K_{ij})$ is defined in the tensor product $\mathcal{H}_F \otimes \mathcal{H}_1$. Hence, there is the problem of the tensor product decomposition on the irreducible representations. The transformed generators look rather cumbersome in terms of the original generators

$$K' = \begin{pmatrix} a & qb \\ -b^\dagger & a^\dagger \end{pmatrix} \begin{pmatrix} \alpha & \gamma^\dagger \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} a^\dagger & -b \\ qb^\dagger & a \end{pmatrix},$$

hence *e.g.* $\varphi(\alpha) = aa^\dagger\alpha + qba^\dagger\gamma + qab^\dagger\gamma^\dagger + q^2bb^\dagger\delta$.

Let us consider the very simple (one-dimensional) irreducible representation of the algebra \mathcal{K} : $\alpha = \delta = 0$, $\gamma \in \mathbf{R}$. The factor \mathcal{H}_1 is one-dimensional and \mathcal{H}_F has to be decomposed into the irreducible representations of \mathcal{K} . To reach this aim one has to find eigenvalues λ and the corresponding eigenvectors $|\lambda\rangle$ of $\alpha' = \varphi(\alpha) = q\gamma(ba^\dagger + ab^\dagger)$ in \mathcal{H}_F . Those of them related by $|\lambda_{n+1}\rangle \simeq \varphi(\gamma)|\lambda_n\rangle$, $\lambda_{n+1} = q^2\lambda_n$ give rise to the invariant subspace of \mathcal{H}_F *w.r.t.* \mathcal{K}' . The Hermitian operator $(ba^\dagger + ab^\dagger)$ is a Jacobian matrix with the entries $q^n c_n$ on the sub-diagonal. Hence the problem of the non-trivial deficiency indices could take place [16].

2.6 A more complicated covariant system is related to the quantum super group $OSp_q(1|2)$ [2]. According to the general arguments of the Introduction, the coaction map gives rise to the extension of the dynamical system and to the representation of the covariantly transformed system in the tensor product with one of the factor being an irreducible representation of the corresponding QG. To find an irreducible unitary representation of the quantum super-group $OSp_q(1|2)$ one has to introduce a $*$ -operation and to analyse the commutation relations among the generators T_{ij} , $i, j = 1, 2, 3$. The matrix T of the $OSp_q(1|2)$ generators is even and has the dimension 3 in the fundamental representation and the grading $(0, 1, 0)$. The compact form of the quadratic relations among the generators is given by the Z_2 -graded FRT-relation (Z_2 -graded tensor product [2, 3])

$$\hat{R}T \otimes T = T \otimes T\hat{R}.$$

The $osp(1|2)$ - R -matrix \hat{R} has the spectral decomposition [2]

$$\hat{R} = qP_5 - q^{-1}P_3 - q^{-2}P_1 ,$$

where the projector indices refer to the dimension $4s + 1$ of the subspaces corresponding to spin 1, $1/2$, 0 (see their explicit expressions in [9]).

Due to the structure of the R -matrix and the orthosymplectic condition $T^{st}C_qT = \gamma C_q$ [2], there are only three independent generators among nine entries of T . One can easily see this from the Gauss decomposition [17] of the matrix T

$$T = T_L T_D T_U ,$$

where T_L , T_U are lower and upper triangular matrices with unities on their diagonal and the diagonal factor $T_D = diag(A, B, C)$. Among the Gauss decomposition generators one finds three independent ones: A , $(T_L)_{21}$ and $(T_U)_{12}$, while the element B is central [17]. Introducing the nine elements of the Gauss decomposition: $T_D = diag(A, B, C)$, $(T_L)_{21,31,32} = (x, y, z)$ and $(T_U)_{12,13,23} = (u, v, w)$ one finds from the FRT-relation [2]:

$$A = T_{11}, \quad x = T_{21}/A, \quad y = x^2/\omega, \quad z = x/q^{1/2},$$

$$u = T_{12}/qA, \quad v = u^2/\omega, \quad w = -q^{1/2}u, \quad B = T_{22} - T_{21}(T_{11})^{-1}T_{12} ,$$

where the elements $B = AC = CA$ are central and $\omega = q^{1/2} - q^{-1/2}$.

Due to the commutativity of T_{13} and T_{31} which are conjugated to each other $T_{31} = -T_{13}^\dagger/q$ according to the *-operation from [2], these elements are diagonal in the Z_2 -graded Fock representation \mathcal{H}_F with the vacuum: $T_{21}|0\rangle = 0$ and the element T_{12} as a creation operator.

From the structure of the quadratic relations among the generators T_{ij} [2] it follows that the four elements T_{12} , T_{32} , T_{13} , T_{31} form a subalgebra of the $OSp_q(1|2)$

$$T_{13}T_{12} = q^{-1}T_{12}T_{13} , \quad T_{13}T_{32} = qT_{32}T_{13} , \quad T_{13}T_{31} = T_{31}T_{13} ,$$

$$T_{12}T_{32} + qT_{32}T_{12} = \lambda q^{1/2}T_{31}T_{13} .$$

Hence, the irreducible representation in the Fock space \mathcal{H}_F is given by T_{12} as creation operator and T_{32} as annihilation operator, while

$$T_{11} = q(T_{12})^2/(\omega T_{13})$$

with T_{13} and T_{31} being diagonal in the basis $|n\rangle \simeq (T_{12})^n|0\rangle$.

Let us now define the quantum OSp -plane, which is an associative super algebra \mathcal{A} with three generators a , ξ , b and the Z_2 -grading $p(a) = p(b) = 0$, $p(\xi) = 1$. Taking into account the similarity of the quantum super-group $OSp_q(1|2)$ to the symplectic group case [1, 7] the defining relations of \mathcal{A} can be written in the R -matrix form with a central extension

$$\hat{R}X \otimes X = qX \otimes X + c_2J , \tag{16}$$

where \hat{R} is the $osp_q(1|2)$ R -matrix, $X = (a, \xi, b)^t$ and the nine component vector

$$J = (0, 0, -q^{-1/2}, 0, 1, 0, q^{1/2}, 0, 0)$$

is the rewritten invariant matrix C_q [2]. One has for the generators the quadratic relations

$$\xi a = a\xi/q, \quad \xi b = qb\xi, \quad [a, b] = \mu\xi^2. \quad (17)$$

where $\mu = q^{1/2} + q^{-1/2} = \lambda/\omega$. The vector J is the eigenvector of the rank one projector P_1 which gives rise to the centrality of the element

$$c_2 = \lambda(q\xi^2/\omega - ab)/q^2(q^{3/2} + q^{-3/2}) = \lambda(\xi^2/q\omega - ba)/q^2(q^{3/2} + q^{-3/2}).$$

This central element is invariant under the $OSp_q(1|2)$ coaction: $X \rightarrow TX$. The algebra \mathcal{A} was identified in [9] as a twisted q -super-oscillator. Although \mathcal{A} has the same irreps as (3) with $b = a^\dagger$ the coaction is more complicated with respect to (11) of the Subsec.2.2 for it includes the number or scaling operator $\xi = \eta q^N$ as well.

3 Conclusion

The problems of the quantum group coaction interpretation and the corresponding tensor product decomposition are especially interesting in the framework of the Poincare group deformation [11, 12, 13]. The corresponding quantum group has many generators and rather complicated quadratic relations among them. Even in the very simple (trivial ?) case when the deformation of the Poincare group is given by the twisting [12] there are two Weyl generators defining the representation. The Hamiltonian of the relativistic system being only covariant under the group transformation law will get extra degrees of freedom after the quantum group coaction [13]. Another kinematical group: the q -Galilei algebra G_q , was connected with the XXZ -model dispersion relation due to the equivalence of the trigonometric function addition law and a non-commutative coproduct [20]. Realizing the generators of G_q in terms of the local spin operators one can obtain by the duality the quantum group coaction. Further interesting possibilities for the representation theory refer to the case when coproduct or coaction maps the original algebra into a tensor product with non-commutative factors (see e.g. [18, 21]).

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References

- [1] L. D. Faddeev, N. Yu. Reshetikhin and L. A. Takhtajan, Alg. Analiz 1 (1989) 178 (Leningrad Math. J. 1 (1990) 193).
- [2] P. P. Kulish and N. Yu. Reshetikhin, Lett. Math. Phys. 18 (1989) 143.
- [3] M. Chaichian and P. P. Kulish, Phys. Lett. B233 (1990), 72.
- [4] M. Chaichian, P. P. Kulish and J. Lukierski, Phys. Lett. B262 (1991) 43.
- [5] W. Pusz and S. L. Woronowicz, Rep. Math. Phys. 27 (1989) 231.
- [6] V. Rittenberg and M. Scheunert, J. Math. Phys. 33 (1992) 436.
- [7] P. P. Kulish, Phys. Lett. A161 (1991) 50.
- [8] P. P. Kulish, Theor. Math. Phys. 85 (1991) 157; i.b. 94 (1993) 193.
- [9] F. Thuillier and J. C. Wallet, Phys. Lett. B323 (1994) 153.
- [10] P. P. Kulish, Proc. of Varenna School on Quantum Groups: Theory and Applications, to be published by Plenum Press, 1995.
- [11] J. Lukierski, A. Nowicki, H. Ruegg and V. Tolstoy, Phys. Lett. B264 (1991) 331.
- [12] M. Chaichian and A. Demichev, J. Math. Phys. 36 (1995) 398.
- [13] J. A. de Azcárraga, P. P. Kulish and F. Ródenas, Phys. Lett. B351 (1995) 123; hep-th/9405161, Fortschr. Phys. (1995).
- [14] M. Arik, In: Symmetries in science VI, Plenum Press, 1993, 47.
- [15] M. Chaichian, H. Grosse and P. Presnajder, J. Phys. A27 (1994) 2045.
- [16] I. M. Burban and A. U. Klimyk, Lett. Math. Phys. 29 (1993) 13.
- [17] E. V. Damaskinsky, P. P. Kulish and M. A. Sokolov, Zap. Nauch. Semin. POMI, 224 (1995) 155; ESI-95-217; q-alg/ 9505001.
- [18] P. P. Kulish and R. Sasaki, Prog. Theor. Phys. 89 (1993) 741.
- [19] P. P. Kulish, Alg. Analiz, 6 (1994) 195.
- [20] F. Bonechi, E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, J. Phys. A 25 (1992) L939; Phys. Rev. B 46 (1992) 5727.
- [21] T. H. Koornwinder, In: Orthogonal polynomials: theory and practice, (ed. P.Nevai) NATO AS Serie 1991, 257.
- [22] O. Babelon and D. Bernard, Commun. Math. Phys. 149 (1992) 279.
- [23] L. Vaksman and Ya. Soibelman, Func. Analiz Pril. 22:3 (1988) 1; Ya. Soibelman, Int. J. Mod. Phys. 7 Supp. 1B (1992) 859.